Duality transformation for O(2) model in d = 2.

Since the vortices are non-local objects in terms of the order parameter field, we will employ a duality transformation under which vortices appear as point-like particles.

The main trick to derive duality is to approximate the cosine in the original H as:

$$e^{K\cos(\nabla\theta)} \simeq e^{K} \sum_{n} e^{-\frac{K}{2}(\nabla\theta - 2\pi n)^2}$$

$$= e^{K} \sum_{n} \int dl \; e^{-\frac{l^2}{2K} + il(\nabla\theta - 2\pi n)}$$

$$= e^{K} \sum_{n} e^{-\frac{n^2}{2K} + in\nabla\theta}$$

where in the last step we have used the identity

$$\sum_{n} e^{i2\pi n x} = \sum_{n=-\infty}^{+\infty} \delta(x-n)$$

The main point of this approximation is that one maintains the periodic nature of cosine, i.e.

$$e^{K\cos(\nabla\theta)} = e^{K\cos(\nabla\theta+2\pi)},$$ without having to directly work with cosine, and instead working with sum of Gaussians, which is technically simpler. Using this approximation, the partition function becomes,

$$Z = \mathrm{Tr}_\theta \prod_{\langle ij\rangle} e^{K\cos(\theta_i-\theta_j)}$$

$$= \mathrm{Tr}_\theta \prod_{\langle ij\rangle} \sum_{n_{ij}} e^{-\frac{n_{ij}^2}{2K} + i n_{ij}(\theta_i-\theta_j)} \qquad (n_{ij}\in\mathbb{Z})$$

(we have dropped an overall constant)

Lets write $n_{ij} = n_\mu(\vec{r})$ where $\mu = \hat{x}/\hat{y}$ denotes the direction of the link $\langle ij\rangle$ and $\vec{r}$ denotes the vertex from which it emanates.

$$Z = \mathrm{Tr}_\theta \prod_{\mu,\vec{r}} \sum_{n_\mu(\vec{r})} e^{-\frac{n_\mu^2(\vec{r})}{2K} + i n_\mu(\vec{r})\nabla_\mu\theta(\vec{r})}$$

Integrating by parts, one may now integrate out θ, to obtain the constraint $\nabla_\mu n_\mu(\vec{r}) = 0$, which may be solved as $n_\mu(\vec{r}) = \varepsilon_{\mu\nu}\partial_\nu a(\vec{r})$. where $a\in\mathbb{Z}$.

$$\Rightarrow \quad Z = \operatorname*{Tr}_{a \in \mathbb{Z}} \; e^{-\sum_{r,\mu} \frac{(\vec{\nabla}_\mu a(\vec{r}))}{2K}}$$

Due to the integer constraint, this model is a bit difficult to work with. One may to relax this constraint is to use the Poisson resummation formula:

$$\sum_{n=-\infty}^{+\infty} f(n) = \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{\infty} d\varphi \; f(\varphi) \, e^{2\pi i m \varphi}$$

Using this,

$$Z = \int Da \sum_{\rho(r) = -\infty}^{+\infty} e^{-\sum_{r,\mu} \frac{(\vec{\nabla}_\mu a)^2}{2K} + \sum_r 2\pi i \, \rho(r) \, a(r)}$$

As we now show that $m(\vec{r})$ actually corresponds to the density of vortices at $\vec{r}$, which act as the source for the spin-waves, which behave like a guage-field. To see this, first notice that the vortex number is given by

$$n = \oint \frac{\vec{\nabla}\theta \cdot dl}{2\pi} = \int \frac{\vec{\nabla} \times \vec{\nabla}\theta}{2\pi} \, d^2x$$

where we have used the Stokes theorem. Note that $\vec{\nabla} \times \vec{\nabla}\theta$ is <u>not</u> zero since $\theta(x)$ is not a smooth f^n near the core of the vortex.

$\Rightarrow$ vortex density $\rho_v(r) = \dfrac{\vec{\nabla} \times \vec{\nabla} \theta}{2\pi}$.

Let's write the original $Z = \mathrm{Tr}_\theta \prod_{r,\mu} e^{K \cos(\nabla_\mu \theta(r))}$

$= \mathrm{Tr}_\theta \prod_{r,\mu} e^{-\frac{K}{2}(\nabla_\mu \theta(r))^2}$ where θ is now allowed to be singular to accomodate vortices. This is equivalent to imposing periodicity on θ. Let's write $\theta = \theta_{smooth} + \theta_{vortex}$ where

$\vec{\nabla} \times \vec{\nabla} \theta_{smooth} = 0$.

$$\Rightarrow Z = \mathrm{Tr}_\theta \int Dn\, e^{-\sum_r \frac{n_\mu(\vec{r})}{2K} + \sum_{r,\mu} i n_\mu [\nabla_\mu \theta_{smooth} + \nabla_\mu \theta_{vortex}]}$$

Integrating out $\theta_{smooth} \Rightarrow \nabla_\mu n_\mu = 0$ which we again solve by $n_\mu = \varepsilon_{\mu\nu} \partial_\nu a$. Note that here n and a are real-valued variables and not integers.

$$\Rightarrow Z = \int Da \int D\theta_{vortex}\, e^{-\sum_r \frac{(\vec{\nabla} a)^2}{2K} + 2\pi i \sum_r a_\mu \frac{(\nabla \times \nabla \theta_{vortex})_\mu}{2\pi}}$$

$$= \int Da \int D\rho_v\, e^{-\sum_r \frac{(\vec{\nabla} a)^2}{2K} + 2\pi i \sum_r a(r)\, \rho_v(r)}$$

Therefore, we identify $\rho(r)$ in

$$Z = \int Da \sum_{\rho(r) = -\infty}^{+\infty} e^{-\sum_{r,\mu} \frac{(\vec{\nabla}_\mu a)^2}{2K} + \sum_r 2\pi i \, \rho(r) \, a(r)}$$

with ρ_{vortex}.

Now one may integrate out the 'guage field' $a(r)$ to obtain a theory only involving the charges (= vortices).

$$Z = \sum_{\rho(r)} e^{-2\pi^2 K \sum_{r,r'} \rho(r) \, G(r-r') \, \rho(r')}$$

where $G(r)$ satisfies $\nabla^2 G(r) = \delta_{r,0}$

$\Rightarrow \quad G(r) = -\frac{1}{2\pi} \log\left(\frac{r}{a}\right) - \frac{1}{4}$ where a is a short-distance cut-off. Therefore, vortices interact with each other via a logarithmic potential, as anticipated from our earlier discussion. Further, since $G(0) \to +\infty$, only configurations with total zero vorticity contribute, i.e.

$$\sum_r \rho(r) = 0. \quad \Rightarrow \quad Z =$$

$$\sum_{\rho(r)} \exp\left[\pi K \sum_{rr'}{}' \rho(r) \log\left(\frac{|r-r'|}{a}\right)\rho(r') + \log(y)\sum_r \rho^2(r)\right]$$

where $\log y = -\frac{1}{4}\pi^2 K$

where the prime in the sum (Σ') denotes that one is restricting to configs with zero net vorticity and constant $\log(y)$ in the second term may be interpreted as a 'chemical potential' for the vortices. Suppose one restricts to vortices with vorticity ± 1. Then the partition f^n may be rewritten as:

$$Z = \sum_{\rho(r)} e^{\pi K \sum'_{rr'} \rho(r) \log|r-r'| \rho(r')} \; y^{N_+ + N_-}$$

where the sum $\sum_{\rho(r)}$ is over distinct locations of the vortices, N_+ = # of +1 vortices, N_- = # of −1 vortices.

At the lowest order in y, the partition f^n recieves contribution from just a pair of vortices, one with vorticity +1 and the other −1.

Z (2 vortex contribution)

$$= y^2 \int \frac{d^2 r_1 \, d^2 r_2}{a^4} e^{-2\pi K \log \left| \frac{r_1 - r_2}{a} \right|}$$

where r_1 and r_2 are locations of the two vortices. The above expression is sufficient to obtain RG eqn. for y at the leading order. Under scaling $a \to ba$ Z changes as

$$Z \to \frac{y^2}{b^4} b^{2\pi K}.$$

Therefore, we need to send $y^2 \to b^{4-2\pi K} y^2$ to keep Z invariant. ⇒

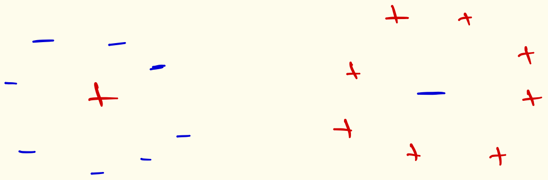
$$y(b) = b^{2-\pi K} y(b=1)$$

$$\Rightarrow \quad \boxed{\frac{dy}{dl} = (2-\pi K) y} \qquad (b = e^{dl})$$

Therefore, when $K > \frac{2}{\pi}$, vortices are irrelevant and we expect that one may rely on the result obtained earlier that didn't take into account compactness of φ ($\varphi \equiv \varphi + 2\pi$).

$$\langle \vec{S}(\vec{r}) \cdot \vec{S}(\vec{0}) \rangle = \left(\frac{a}{r}\right)^{\frac{1}{2\pi K}}$$

On the other hand, when $K < \frac{2}{\pi}$, vortices are relevant. To figure out the full RG flow and phase diagram, we need an RG equation for K. We will first derive it heuristically, and then return to a more detailed argument later. Since K controls the strength of interaction between two vortices, it is the dielectric constant of this medium where vortices are charges. If two vortices are separated by some distance $r \gg a$, then the force between them will be renormalized due to screening by other charges in between.



Under RG one integrates out short-distance modes. Therefore, RG process will <u>decrease</u> the screening as it will reduce the amount of intervening charges. Less screening $\Rightarrow$ smaller dielectric constant $\Rightarrow$ larger K. Further, this effect should be an even f^n of vortex fugacity 'y' since $\pm$ vortices appear in pairs. Therefore, at the leading order

$$\frac{dK}{dl} = -A' y^2$$

where A' is some positive constant. It is customary to define $x = 2 - \pi K$, so that the two coupled RG equations are:

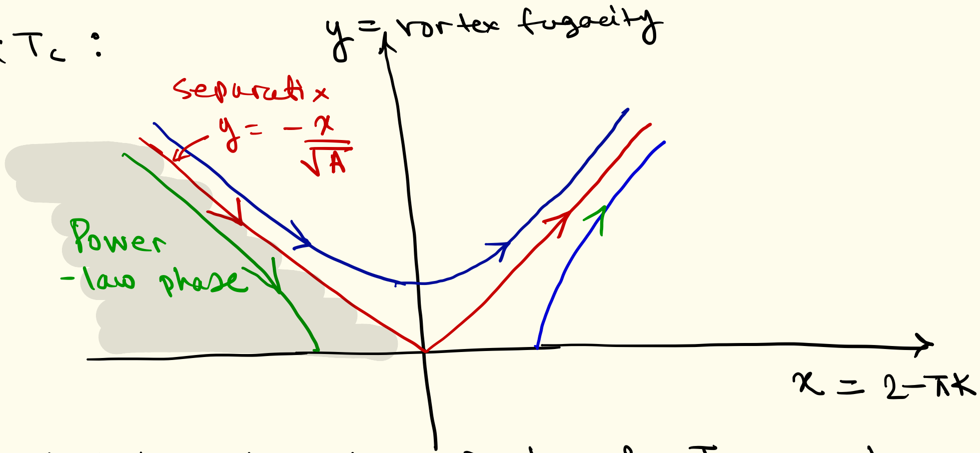
$$\boxed{\begin{aligned} \frac{dy}{dl} &= xy \\ \frac{dx}{dl} &= A y^2 \end{aligned}}$$

$(A = \pi A' > 0)$

Solution of RG Eqns and Physical consequences:

First, notice that $\dfrac{d(x^2 - Ay^2)}{dl} = 0$.

Therefore RG trajectories are hyperbolas in the (x,y) plane. Further, the lines $x = \pm\sqrt{A}\,y$ pass through the origin. The significance of $y = -\dfrac{x}{\sqrt{A}}$ is that it is a separatrix which separates the behavior for $T > T_c$ from $T < T_c$:



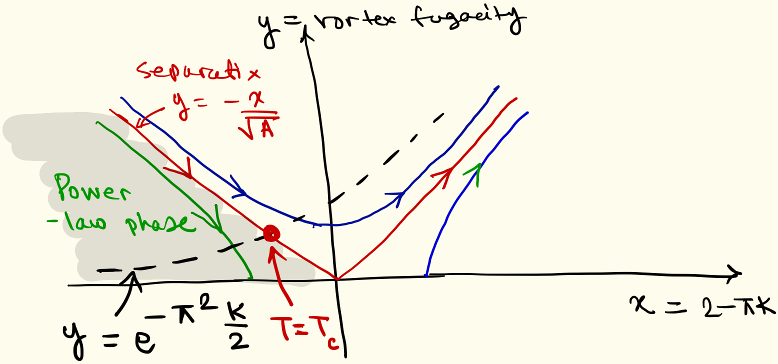
The shaded region above is the low-T power-law phase because the vortex fugacity flows to zero in that region. Therefore to determine the K_c where the system undergoes the transition from the power-law correlations to exponentially decaying ones, one needs to solve,

$$-x = \sqrt{A}\, y$$

Since $y \sim e^{-\frac{\pi^2 K}{2}}$ and $-x = \pi K - 2$,

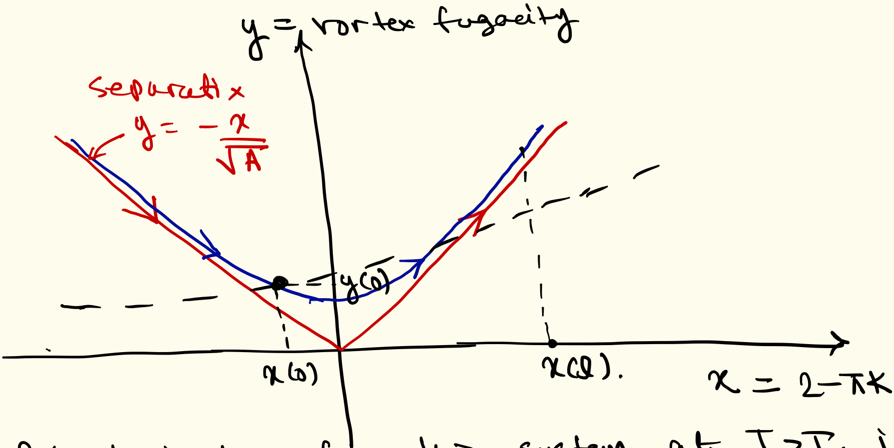
K_c satisfies $K_c = \dfrac{2 + \sqrt{A}\, e^{-\frac{\pi^2 K_c}{2}}}{\pi}$.

Therefore, we find a small correction to our earlier estimate $K_c = \frac{2}{\pi}$. Pictorially:



Correlation length close to T_c:

For $T < T_c$, the correlation length is infinity since one is in the power-law phase. As one approaches from $T > T_c$, the correlation length diverges as $T \to T_c$. This length scale may be thought of as the Debye screening length since for $T > T_c$, vortices can move around freely and one is in the plasma phase. Due to screening, they do not interact via logarithmic potential in this phase and the effective interaction is short-ranged which defines the Debye screening length, which diverges as $T \to T_c$, signalling the breakdown of screening. To find this divergence, consider an RG trajectory just above T_c:



The RG trajectory for this system at $T > T_c$ is

$$x^2(l) - A\,y^2(l) = x^2(0) - A\,y^2(0).$$

Since one is close to T_c, we write $x(0) = -\sqrt{A}\,y(0)[1+t]$ where $t = \frac{T - T_c}{T_c} \ll 1$.

when $l = 0$, we assume that the correlation length $\xi = \xi_0 = O(1)$. $\Rightarrow$ $\xi(l) = \xi_0 e^{l}$.

Integrating $\frac{dx}{dl} = Ay^2$ $\Rightarrow$

$$l = \int_0^l dl' = \int_{x(0)}^{x(l)} \frac{dx}{Ay^2} = \int_{x(0)}^{x(l)} \frac{dx}{x^2 - [x^2(0) - Ay^2(0)]}$$

$$\simeq \int_{x(0)}^{x(l)} \frac{dx}{x^2 + 2Ay^2(0)t}.$$

The dominant contribution comes when $x^2 \in [0, 2Ay^2(0)t]$.

$$\Rightarrow \quad l \sim \frac{1}{y(0)\sqrt{t}} \quad \Rightarrow \quad \xi(l) \sim \xi_0 e^{c/y(0)\sqrt{t}}$$

where c is some constant. Therefore, the correlation length diverges faster than any power law of t.

Specific heat:

The singular part of the free energy scales as $b^{-d} \sim e^{-2l} \sim \xi^{-2} \sim e^{-1/\sqrt{t}}$. Therefore, instead of a power-law singularity, one obtains an essential singularity i.e. all derivatives

of free energy are continuous across T_c.
Therefore, this could be called an 'infinite order' phase transition.

Stiffness and its jump across T_c :